Lesson 1. Logic (Part I)

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Contents

What is logic?

What is logic?

- The formal mathematical study of the methods, structure, and validity of mathematical deduction and proof.[1]
- The principles underlying the units in a computer system that produce results from data, Greek *logikos* concerning speech or reasoning [2].

[1] Mathworld

[2] Collins dictionary

Propositions

Definition

A **proposition** is a sentence or astatement that can be said without ambiguity, and it can be either true or false, exclusively. The **logic value** of a proposition is 1 if it is true and 0 if it is false.

Sometimes 1 and 0 are replaced by V and F.

Some examples

(1) *"Aristotle was a woman"* is a false proposition (logic value equal to 0).

(2) *"Where are you tonight?"* is not a proposition (it is a question).

(3) *"If a number is a power of 10, then it is even"* is a true proposition (logical value equal to 1).

(4) "$x + 4 = 0$" is not a proposition (it depends on which is the value of "x").

Propositions

What does happen with Dr. House famous quotation?

Everybody lies

This is a modern version of Epimenides Paradox

He was Cretan and said:

"All Cretans are liars."

Are they true or false?

Propositions

Is this statement a proposition

This sentence is false

- If it was true, the statement itself says that it is false, so it should be true (A contradiction).
- If it was false, the statement would be true, so in fact it should be false (A contradiction).

This happens when we are using the metalinguistic function of the language. Situations like this happen even if we consider a set of axioms (Gödel).

We will deal only with propositions, in order to avoid these situations.

We will denote them by capital letters $P, Q, R, S, \ldots$.

Logic conectives

Logic connectives are symbols that let us construct composed propositions from simpler ones.
If P and Q represent generic propositions, the most frequently used logic connectives are the following:

Connective	Symbol	It is written as	It can be read as
Negation	$\neg$	$\neg P$	not P
Conjunction	$\wedge$	$P \wedge Q$	P and Q
Disjunction	$\vee$	$P \vee Q$	P or Q
Implication	$\rightarrow$	$P \rightarrow Q$	if P, then, Q
			P only if Q
Bi-implication	$\leftrightarrow$	$P \leftrightarrow Q$	P if, and only if, Q

Meaning of the connectives

The meaning of the logic connectives is given by the following tables.

- Negation: $\neg P$ is true when P is false, and it is false when P is true. Its truth table is

P	$\neg P$
1	0
0	1

- Conjunction: $P \wedge Q$ is true when P and Q are simultaneously true. It is false in the other cases. The corresponding truth table is

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Meaning of the connectives

- Disjunction: $P \vee Q$ is true when one of the propositions P or Q is true, and false if both P and Q are false.
  Its truth table is

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

We understand (in this course) that the disjunction will not be exclusive!

Meaning of the connectives

- Conditional: $P \to Q$ is only false when Q is true and Q is false, and it is true in the rest of the cases.
  Usually, P is known as the **hypothesis** and Q as the **consequence**. Its truth table is

P	Q	$P \to Q$
1	1	1
1	0	0
0	1	1
0	0	1

  This means that Q is true when P is true, but it says nothing about the truth value of Q when P is false.

Meaning of the connectives

- Bi-conditional: $P \leftrightarrow Q$ is true when both propositions are true or both of them are false. It is false in the other cases.
  Its corresponding truth table is

P	Q	$P \leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

  It is equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$

Prevalence of connectives

The ambiguities are avoid if we stablish hierarchy in the connectives by using connectives.

Hierarchy (up to down)

- $\leftrightarrow$
- $\rightarrow$
- $\wedge$, $\vee$
- $\neg$

The priority of a connective is higher than the other means that, in the absence of parentheses, the second one is consider before the first.

Examples

- The proposition $P \wedge Q \rightarrow R$ is right since it has no misunderstanding. According to the connectives hierarchy, it is equivalent to $(P \wedge Q) \rightarrow R$.
- $P \wedge Q \vee R$ is not properly written, since the connectives $\wedge$ and $\vee$ has the same hierarchy category. Parentheses are needed to avoid vagueness: either $(P \wedge Q) \vee R$ or $P \wedge (Q \vee R)$ are valid, depending on the meaning.

Propositional functions

One of the most significatives trends of logic is that its validity exclusively depends on its "form", and not on the meaning of the underlying propositions.

Definition (Propositional formula)

A **propositional formula** (or **propositional function**, or **propositional form**) is any expression formed by

(a) symbols (usually letters) that represent other propositional formulas, which are called *propositional variables*,

(b) logic connectives, and

(c) pairs of parentheses $(\ldots)$,

which is built using the following rule: if $\mathscr{A}$ and $\mathscr{B}$ are propositional forms, then the following expressions are propositional forms too:

$$(\neg\mathscr{A}),\quad (\mathscr{A}\wedge\mathscr{B}),\quad (\mathscr{A}\vee\mathscr{B}),\quad (\mathscr{A}\rightarrow\mathscr{B}),\quad \& \quad (\mathscr{A}\leftrightarrow\mathscr{B}).$$

Usualy parentheses are removed if doing so, there is no ambiguity (according to the hierarchy of logic connectives).

Examples

Examples

- $\wedge P \neg \vee \neg Q$ is not a propostional formula (some reasons: the wedge should appear between formulas, we cannot use $\neg$ in front of $\vee$, and the prevalence of $\wedge$ and $\vee$ is not clear).
- $(P \wedge (\neg Q) \rightarrow R)$ is a propositional formula because it follows the construction rules. Unnecessary parentheses can be removed: $P \wedge \neg Q \rightarrow R$.
- In a propositional formula, **variables** can be replaced by **propositions** making a proposition.

  This proposition will ceratinly have a logic value. For instance, if in $P \wedge \neg Q \rightarrow R$ we take P :="Pepe goes to the cinema", Q :="Jaimito has flu', and R :="The sun is a star", then we get that this proposition has 1 as logic value (since the consequence is true, the sun is a star).

Tautologies and contradictions

Definition

- A **tautology** is a propositional formula which is *always true* (independently of the logic values of the propositions that form it). A tautology is denoted by τ.
- A **contradiction** is a propositional formula that it is *always false* (independently of the logical values of the variables that form it). A contradiction will be denoted by ϕ.
- A **contingency** is a propositional formula which is not a tautology nor a contradiction.

Example

- $\neg(P \wedge Q \rightarrow R)$ is a contingence.
- $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$ is a tautology.
- $P \wedge \neg P$ is a contradiction.

Logic equivalence

Definition (Equivalent propositional formulas)

Two propositional formulas are **equivalent** if they have the same logic values for every set of logic values of its components.
In other words, two propositional formulas P and Q are equivalent if $P \leftrightarrow Q$ is a tautology.
This can be denoted either as $P \equiv Q$ or as $P \leftrightarrow Q$.

Example

The propositional formulas $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are equivalent. (This is known as one of De Morgan's laws, that we will see later on).

Logic implication

Definition (Logic implication)

Given two propositional formulas P and Q, it is said that P **implies** Q, if whenever P is true, Q so it is.

In other words, the conditional expresion $P \rightarrow Q$ is a tautology.

We write it either as $P \vdash Q$ or $P \rightarrow Q$.

It is said that P is the **antecedent** and Q is the **consequent** of the implication.

Propositional forms algebra

Boolean properties.

1. Associative properties

   $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
   $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

2. Commutative properties

   $P \vee Q \equiv Q \vee P$
   $P \wedge Q \equiv Q \wedge P$

3. Distributive properties

   $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

4. Identity element for $\vee$ and $\wedge$

   $P \vee \phi \equiv P$
   $P \wedge \tau \equiv P$

5. An element and its inverse for $\vee$ and $\wedge$

   $P \vee \neg P \equiv \tau$
   $P \wedge \neg P \equiv \phi$

Propositional forms algebra

Some properties concerning disjunction, conjunction, and negation.

- Absorption property

  $\tau \vee P \equiv \tau, \quad \phi \wedge P \equiv \phi$

- Simplification properties

  $P \vee (P \wedge Q) \equiv P, \quad P \wedge (P \vee Q) \equiv P$

- Idempotent law

  $P \vee P \equiv P$
  $P \wedge P \equiv P$

- De Morgan's laws

  $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
  $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

- Double negation law or involution law

  $\neg(\neg P) \equiv P$

Propositional forms property

Properties involving conditionals, biconditionals, and disjunctions.

- Conditional-disjunction equivalence

  $$P \to Q \equiv \neg P \vee Q$$

- Conditional-biconditional equivalence

  $$(P \to Q) \wedge (Q \to P) \equiv P \leftrightarrow Q$$

- Transposition property

  $$P \to Q \equiv \neg Q \to \neg P$$

- Exportation law

  $$(P \wedge Q) \to R \equiv P \to (Q \to R)$$

Simplification of the following propositonal forms

All previous equivalences can be used in order to *simplify* a certain propositional form, that is, to obtain an equivalent simpler propositional form.

Let us go to simplify the following propositional form

$$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$$

Simplification of the following propositonal forms

$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$	
$\Leftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$	Distributive law
$\Leftrightarrow p \vee [(q \vee r) \wedge (\neg t \vee r) \wedge (t \vee \neg q)]$	Commutative property
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)]$	Distributive property
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (\neg\neg t \vee \neg q)]$	Double negation property
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)]$	De Morgan's law
$\Leftrightarrow p \vee [((\neg t \wedge q) \vee r) \wedge \neg(\neg t \wedge q)]$	Commutative property
$\Leftrightarrow p \vee [((\neg t \wedge q) \wedge \neg(\neg t \wedge q)) \vee (r \wedge \neg(\neg t \wedge q))]$	Distributive law
$\Leftrightarrow p \vee [\phi \vee (r \wedge \neg(\neg t \wedge q))]$	Idempotent law
$\Leftrightarrow p \vee [r \wedge \neg(\neg t \wedge q)]$	Identity element
$\Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$	De Morgan's law

Therefore $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) \equiv p \vee [r \wedge (t \vee \neg q)]$

Inference

Definition (Inference)

We call **inference** or **proof** to the process that, starting from some **hypothesis** $H_1, H_2, \ldots, H_n$ (that is, some propositions which are assumed to be true), we can arrive to a **conclusion** C.
In other words, what we do is to **prove** the logical implication $H_1 \wedge H_2 \wedge \cdots \wedge H_n \rightarrow C$ with a finite sequence of steps which consists on using the inference laws (that we are going to see).

The most frequent inference processes are the following:

- Direct inference
- Conditional inference
- Biconditional inference
- Inference by contradiction

Direct inference (Direct proof)

Direct inference

The direct inference consists on deducing the conclusion from the hypothesis directly using some rules known as **inference laws**.

They are the following:

1. **Use of logical equivalences**
   Every propositional form can be replaced by an equivalent one.
2. **Use of the hypothesis nad inference rules**
   Every hypothesis or every formula obtained in the proof can be used with an inference rule at any step of the proof.

The most usual inference laws

1. **Conjunction**
   If at some step we have the propositional form P and we have also Q at another step, then we can introduce the propositional form $P \wedge Q$.
   - $\{P, Q\} \vdash P \wedge Q$
2. **Simplification**
   If we have the conjunction of two propositional forms, then we can take out one of them.
   - $\{P \wedge Q\} \vdash P$
   - $\{P \wedge Q\} \vdash Q$
3. **Addition**
   If we have a propostional form, then we can add another propositional form.
   - $\{P\} \vdash P \vee Q$
   - $\{Q\} \vdash P \vee Q$

Modus Ponens and Modus Tollens

4. **Modus ponendo ponens (or modus ponens).** This means " *the way that affirms by affirming*".
   If we have an implication and the antecedent is true, then we can get the propositional formula in the consequent.
   - $\{P, P \rightarrow Q\} \vdash Q$

5. **Modus tollendo tollens (or modus tollens).** This means *"the way that denies by denying"*.
   If we have an implication and we have the negation of the consequent, then we can get the negation of the propositional formula in the antecedent.
   - $\{\neg Q, P \rightarrow Q\} \vdash \neg P$

6. **Disjunctive syllogism**
   If we have a disjunction and the negation of one of the propositional forms, then we can affirm the other propositional form.
   - $\{\neg P, P \vee Q\} \vdash Q$
7. **Hypothetical syllogism**
   If the consequent of one implication is the antecedent of another one, then we can connect them.
   - $\{P \rightarrow Q, Q \rightarrow R\} \vdash (P \rightarrow R)$

8. The following rules are used to get a conclusion from two implications.
   - $\{P \rightarrow R, Q \rightarrow S\} \vdash (P \wedge Q) \rightarrow (R \wedge S)$
   - $\{P \rightarrow R, Q \rightarrow S\} \vdash (P \vee Q) \rightarrow (R \vee S)$

9. If two disjunctions are true and one propositional formula is affirmed in one and denied in the other, then we can get out the other propositional formula.
   - $\{P \vee Q, \neg P \vee Q\} \vdash Q$

Example

Prove that from the following hypothesis

H1: $P \rightarrow Q$
H2: $P \leftrightarrow \neg R$
H3: $\neg Q$

can be deduced the conclusion R.

H1:	$P \rightarrow Q$	
H2:	$P \leftrightarrow \neg R$	
H3:	$\neg Q$	
4:	$\neg P$	Modus tollens (1,3)
5:	$(P \rightarrow \neg R) \wedge (\neg R \rightarrow P)$	Conditional-biconditional equivalence (2)
6:	$\neg R \rightarrow P$	Simplification (5)
7:	$\neg\neg R$	Modus tollens (4,6)
8:	R	Double negation (7)

Conditional inference

Definition (Conditional inference)

Conditional inference consist on proving an implication $Q \to R$ from several hypothesis $H_1, H_2, \ldots, H_n$.

It is based on the equivalence

$$H \to (Q \to R) \equiv H \wedge Q \to R.$$

How to prove it?

In order to prove $Q \to R$, we will proceed as follows :

1. we will add Q to the set of hypothesis as an additional hypothesis, and
2. we will deduce R from Q and the other hypothesis by direct inference.
3. We will introduce the form $Q \to R$ as a final step, indicating that it has been obtained by direct proof.

Example

We will prove that the hypothesis

H1: $U \rightarrow R$
H2: $R \wedge S \rightarrow P \vee T$
H3: $Q \rightarrow U \wedge S$
H4: $\neg T$

let us conclude $Q \rightarrow P$.

H1:	$U \rightarrow R$		
H2:	$R \wedge S \rightarrow P \vee T$		
H3:	$Q \rightarrow U \wedge S$		
H4:	$\neg T$		
5:		Q	We introduce an additional hypothesis
6:		$U \wedge S$	Modus ponens (3,5)
7:		U	Simplification (6)
8:		R	Modus ponens (1,7)
9:		S	Simplification (6)
10:		$R \wedge S$	Conjunction (8,9)
11:		$P \vee T$	Modus ponens (2,10)
12:		P	Disjunctive syllogism (4,11)
13:	$Q \rightarrow P$		Direct inference (direct proof), steps 5 to 12.

Bi-conditional inference

Definition (Bi-conditional inference)

Bi-conditional inference is used when the conclusion is of the type $P \leftrightarrow Q$.

It is based on the equivalence between condicional and bicondicional:

$$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P).$$

How to prove it?

When we have to prove a bi-conditional $P \leftrightarrow Q$ we will proceed as follows:

1. We prove that the hypothesis let us deduce the conditional $P \to Q$ (direct conditional), and
2. we also prove $Q \to P$ (converse conditional).

Inference by contradiction

Definition (Inference by contradiction)

This method consists on **assuming as a new hypothesis the negation of the conlusion**. Then after applying inference and equivalence rules **we arrive to a propositional form and its negation**. Since this is a contradiction, our additional hypothesis cannot be fulfilled, and its contrary must be true.

It is based on the equivalence $(P \wedge \neg Q) \rightarrow \phi \equiv P \rightarrow Q$.

How to prove $P \rightarrow Q$ by contradiction?

1. We assume the contrary, which is $P \wedge \neg Q$ (by equivalence rules).
2. We apply inference and equivalent rules until we get R and $\neg R$ for some propositional form R (R can be P, Q or any other propositional form).
3. We say that we arrive to a contradiction, and we affirm that $P \rightarrow Q$ is true.

Remark

In general, if we want to prove a propositional form S, what we do is to introduce $\neg S$ as an additional hypothesis, and we try to get a contradiction