Lesson 1. Logic (Part II)

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Contents

Motivation of the predicate calculus

Propositional calculus is not enough to justify arguments of the following type:

All multiples of 4 are divisible by 2.
24 is a multiple of 4.
Therefore, 24 is divisible by 2.

This is because it is not possible to express that these two different propositions have some elements in common.

The common elements are **"to be divisible by 2"** and **"to be multiple of 4"**.

Besides, it does not fit to be applied to specific objects and to groups of objects.

*"***all** *multiples of 4"* or *"***some** *multiples of 4".*

Therefore, we will introduce the **predicate calculus** to solve these problems.

Predicates

Definition (Predicate)

A **predicate** describes a property of one or several objects.

Examples of predicates

(a) … is red.

(b) … has a long nose.

(c) … is a multiple of 4.

(d) … is divisible by 2.

The … can be **filled** with names of suitable objects in order to build a proposition. This new proposition can be either true or false in the sense of propositional calculus.

Examples

(a) **The car** is red.

(b) **John** has a long nose.

(c) **24** is a multiple of 4.

(d) **64** is divible by 2.

Notation

We will usually use capital letters to refer to predicates:

$R(\ldots) =$ "… is red".
$N(\ldots) =$ "… has a long nose".
$M(\ldots) =$ "… is a multiple of 4".

We will use lowercase letters to denote specific objects or individuals:

$c =$ "This car"
$j =$ "John"

The previous examples read as follows

$R(c) =$ "This car is red".
$N(j) =$ "John has a long nose".
$M(24) =$ "24 is a multiple of 4". (Here 24 refers to itself as an element)

About the values of truth in predicate logic

$R(x)$ is not, itself, a proposition because it cannot be declared as true or false. However, this can be done if we replace the variable **x** by a specific object or person.

Universal quantifier

Definition (Universe)

We call **universe** to the class of objects considered for certain predicate.

Example (Consider the universe of all animals for the following predicates)

$S(x) =$ "x is a sheep".
$W(x) =$ "x is white".

Let us consider the following statement: "All sheeps are white".

We can rewrite this statement as: **"For all x, if x is a sheep, then x is white"**.

Definition (Universal quantifier)

The symbol $\forall$ is called the **universal quantifier** and $\forall x$ means **"for all x"**.

Using the universal quantifier, the previous statement can be represented as:

$$\forall x\,(S(x) \rightarrow W(x)) \text{ or as } \forall x\, S(x) \rightarrow W(x).$$

Existential quantifier

Let us consider again the universe of all animals and the previous predicates: $S(x) =$ "x is a sheep" and $W(x) =$ "x is white".

Consider the following statement

"Some sheeps are not white".

We can rewrite this statement as follows:

"There is some x such that x is a sheep and x is not white".

Definition (Existential quantifier)

The symbol $\exists$ is called the **existential quantifier** and $\exists x$ means **"there is some x"** or **"there exists some x"**.

Using the existential quantifier, the previous statement can be represented as:

$$\exists x\,(O(x) \wedge \neg B(x)) \text{ or as } \exists x\, O(x) \wedge \neg B(x).$$

Propositional functions

A predicate can have more than one variable!

In the universe of integers we can consider:

The predicate $T(x, y) :=$ "x is a multiple of y",
and $T(10, 5)$ will be the proposition *"10 is a multiple of 5"*.

Definition (Propositional function)

A **propositional function** in predicate calculus is a expression of the form $P(x_1, x_2, \ldots, x_n)$, where P is a predicate and $x_1, x_2, \ldots, x_n$ are the variables.

Equivalences and inference laws for predicate calculus

The inference process in the predicate calculus is similar to the one in propositional calculus. Nevertheless, we add some specific equivalences and inference laws where quantifiers appear.

Negation of quantifiers

- $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

Disjunctions and conjunctions with quantifiers

- $\forall x\, P(x) \wedge Q(x) \equiv \forall x\, P(x) \wedge \forall y\, Q(y)$
- $\exists x\, P(x) \vee Q(x) \equiv \exists x\, P(x) \vee \exists y\, Q(y)$
- $\forall x\, P(x) \vee \forall y\, Q(y) \vdash \forall x\, P(x) \vee Q(x)$
- $\exists x\, P(x) \wedge Q(x) \vdash \exists x\, P(x) \wedge \exists y\, Q(y)$

Be careful with the last four rules: **The first two are equivalences, the other two are inferences** (the part on the left is stronger than the part on the right.

Inference laws for predicate calculus

Universal specification

From the predicate $\forall x\, P(x)$ we can deduce $P(a)$ for **any** element a of the universe. This element is said to be **arbitrary**.

Existential specification

From $\exists x\, P(x)$ we can deduce $P(a)$ for **some specific** element a of the universe.

Universal generalization

If $P(y)$ is true **for any** element y of the universe (that is for any arbitary element), then we can deduce $\forall x\, P(x)$.

Existential generalization

If $P(a)$ is true for **certain** element a of the universe, then we can deduce $\exists x\, P(x)$.

Example

Let us see that the following argument is right:

Some soccer supporters are also basketball supporters.
Soccer supporters don't go to the cinema on Sunday afternoon.
Therefore, there are some basketball supporters who don't go to the cinema on Sunday afternoon

Let us define the following predicates

$S(x) :=$ "x is a soccer supporter"
$B(x) :=$ "x is a basketball supporter"
$C(x) :=$ "x goes to the cinema on Sunday afternoon"

These are the hypothesis:

- $\exists x\, S(x) \wedge B(x)$
- $\forall x\, S(x) \rightarrow \neg C(x)$

and the conclusion will be $\exists x\, B(x) \wedge \neg C(x)$.

Example

Let us see the proof:

H1:	$\exists x\, S(x) \wedge B(x)$	
H2:	$\forall x\, S(x) \rightarrow \neg C(x)$	
3:	$S(a) \wedge B(a)$	Existential specification of (1) for certain $x = a$ specific.
4:	$S(a) \rightarrow \neg C(a)$	Universal specification of (2) for the $x = a$ specified in (1).
5:	$S(a)$	Simplification (3)
6:	$\neg C(a)$	Modus ponens (4,5)
7:	$B(a)$	Simplification (3)
8:	$B(a) \wedge \neg C(a)$	Conjunction (7,6)
C:	$\exists x\, B(x) \wedge \neg C(x)$	Existential generalization (8)

Axiomatic method

In the next sextion we set some terminology that is commonly used in mathematics.

- Mathematics use the **axiomatic method**, that is, from some mathematical definitions and certain statements that are assumed to be true (namely **axioms**), it can be proved (using equivalences and inference rules) other statements that make up the theory. The inference processes are called **proofs**.
- The statements deduced from the axioms and the definitions are called **theorems**.

Theorems, propositions, lemmas (or lemmata) an corollaries

The word **theorem** is usually used just to refer to the **main statements** that are proved, and the word **propositions** to others which are not so important. However, other words are also used: A **corollary** is a direct (or almost direct) consequence of a certain theorem. A **lemma** is a previous auxiliary (usually quite technical) statement that is used in the proof of a theorem.

Theorems

Usually all the theories deal with predicate calculus. Propositional calculus is less powerful.

Some definitions

- A **theorem** is usually a formula of the form $P \rightarrow Q$ that is a tautology.
- The conditions in the formula P are usually called **hypothesis** and the conclusion Q is called **thesis**. The **proof** is the inference process which let us deduce Q from P.
- Usually, $P \rightarrow Q$ is called the **direct implication**, and when we change the place of P and Q, we called the implication $Q \rightarrow P$ as the **converse implication**. This terminology is usually used to refer to the both implications that appear in a biconditional.

Remember

The formula $P \rightarrow Q$ is **equivalent to** $\neg Q \rightarrow \neg P$, **not to** $Q \rightarrow P$.

Peano Axioms

The set $\mathbb{N}$ is characterized using the **Peano Axioms**:

Peano Axioms

1. There exists a natural number called 1. (Some people also include also the 0)
2. Every natural number n has a **successor** that we will represent by $n+1$.
3. Whenever two natural numbers are different, then their successors will be different.
4. Every natural number, except 0, is the successor of another natural number.
5. **Induction principle:** If A is a set of natural numbers such that
    - 0 belongs to A, and
    - whenever n belongs to A, then its successor $n+1$ also belongs to A,

    then $A = \mathbb{N}$.

Peano Axioms

Remarks

- The induction principle can be applied to proof statements concerning the natural numbers explicitly (such that $\forall n \in \mathbb{N}\ n \leq 2^n$) or implicitly ($\forall n \in \mathbb{N}\ n$ straight lines in general position divide a plane into $n+1$ regions).
- The role of 0 can also be played by any natural number n_0. In this case A would contain all natural numbers from n_0 onwards.
- There exists a number of versions of it (weak induction, induction on two variables,...). We will only deal with the one presented before.
- Using predicate logic, the induction principle reads as follows: If $P(n)$ is a predicate in the universe of natural numbers, then

$$P(1) \wedge (\forall n\, P(n) \rightarrow P(n+1)) \rightarrow \forall n\, P(n)$$

Example (The induction principle can be rewritten as)

H1: $P(1)$

H2: $\forall n\, P(n) \rightarrow P(n+1)$

C: $\forall n\, P(n)$

Since it is an axiom we assume it as true, and it need not to be proved.