Lesson 2.1 Sets

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Contents

Definition of set

Definition (Set)

A **set** is a well-defined collection of objects. The objects in the set are called **elements** of the set.

NOTATION:

- The sets are denoted with capital letters: $A, B, C, \ldots$
- The (generic) elements are denoted with minuscule letters.
- If C is a set and a is an element of C, $a \in C$ means *"a belongs to C"* and $a \notin C$ means *"a does not belong to C"*.

Definition (Equality of sets)

Two sets A and B are said to be **equal** if they have the same elements. It is denoted by $A = B$.

Definition (Empty set)

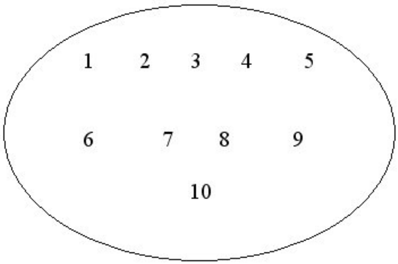
The **empty set** ($\emptyset$) is the one which has no elements.

How to define a set?

- **Extensional definition**: showing, between brackets, all the elements separated by commas.
  Example:
  $$A = \{1,2,3,4,5,6,7,8,9,10\}.$$
  In this case, the order the elements are written in the set lacks of importance.
- **Intensional definition,**: Stating a property that characterizes all the elments of the set. For example, the aforementioned set A can be defined as follows:
  $$A = \{n \in \mathbb{N} \mid 1 \leq n \leq 10\}.$$
  $\mathbb{N}$ is the usual way to denote **the set of all natural numbers**.
  It can be read as *"A is the set of all elements x belonging to* $\mathbb{N}$ **such that** *they are greater or equal than 1 and less or equal than 10"*. The symbol $|$ used to represent "such that", is sometimes replaced by two points:
  $$A = \{x \in \mathbb{N} : 1 \leq x \leq 10\}.$$

Venn diagrams

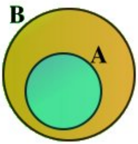
A useful way to visualize properties of the sets is using **Venn diagrams**:



Subsets

Definition (Subset)

Given two subsets A and B, it is said that A is a **subset** of B or that A **is contained in** B if all the elements of A also belong to B. It is denoted as $A \subseteq B$.



- Given a set C, $\emptyset$ and C are always two subsets of C.
- Given two subsets A and B, we have

$$A = B \longleftrightarrow A \subseteq B \text{ and } B \subseteq A$$

Power set

Definition (Power set)

If C is any subset, the **power set of** C, denoted by $\mathscr{P}(C)$, is the set whose elements are all the subsets of C. That is:

$$\mathscr{P}(C) = \{A : A \subseteq C\}.$$

Example

- $\mathscr{P}(\emptyset) = \{\emptyset\}$
- Si $A = \{a, b, c\}$:

$$\mathscr{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$$

Complement of a set

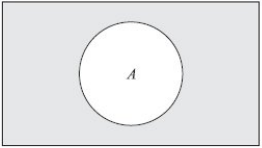
We assume that all the sets that we are considering are subsets of a **universal set** called E (we will denote it in the diagrams by a rectangle).

Definition (Complement of a set)

Given a subset $A \subseteq E$ we called the **complement of a set** A to the set:

$$A^c = \{x \in E \mid x \notin A\},$$

that is, it is the set that consists on all the elements of the set E which do not belong to A (this is the dark section in the Venn diagram).



Complement of a set

Properties

- $E^c = \emptyset$
- $\emptyset^c = E$
- $A = B \Leftrightarrow A^c = B^c$
- $B \subseteq A \leftrightarrow A^c \subseteq B^c$

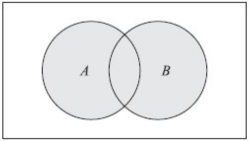
Union of sets

Definition (Union)

Given two subsets $A, B \subseteq E$ we called **union of** A **and** B to the set:

$$A \cup B = \{x \in E \mid x \in A \ \vee \ x \in B\},$$

that is, it is the set of all elements of the universal set E that either belong to A or to B (dark zone in the Venn diagram).



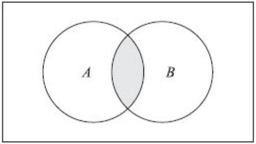
Intersection of sets

Definition (Intersection)

Given two sets $A, B \subseteq E$ it is called **intersection of A and B** to the set:

$$A \cap B = \{x \in E \mid x \in A \ \wedge \ x \in B\},$$

that is, it is the set of all elements of E that belong to A and to B at the same time (dark zone in the Venn diagram)



Properties

Properties

- $A \subseteq A \cup B$
- $A \cap B \subseteq A$
- $A \subseteq B \Leftrightarrow A \cap B = A \leftrightarrow A \cup B = B$

Definition (Disjoint)

Given two sets $A, B \subseteq E$ it is said that they are **disjoint** if $A \cap B = \emptyset$.

Example

If $\mathbb{Q}$ and $\mathbb{I}$ denote, the set of rational and irrational real numbers, then $\mathbb{Q} \cap \mathbb{I} = \emptyset$. So that they are disjoint.

Generalization of the union and the intersection

These notions can be generalized to an arbitrary amount of sets:

Example

If for every real positive number $a > 0$ we define $B_a = [-a, a] \subseteq \mathbb{R}$, we have

$$\bigcup_{a>0} B_a = \mathbb{R} \text{ and } \bigcap_{a>0} B_a = \{0\}.$$

If $\mathscr{A}$ consists on a finite number of sets $A_1, \ldots, A_n$, we denote its union and intersection by $A_1 \cup A_2 \cup \cdots \cup A_n$ and $A_1 \cap A_2 \cap \cdots \cap A_n$ respectively.

Properties

Let $A, B, C \subseteq E$.

1. Associative properties
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$
2. Commutative properties
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
3. Distributive properties
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
4. Identity element
$$A \cup \emptyset = A$$
$$A \cap E = A$$
5. Inverse element
$$A \cup A^c = E$$
$$A \cap A^c = \emptyset$$

Sets vs. propositional forms

1. Associative properties

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

2. Commutative properties

$$A \cup B = B \cup A$$
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4. Identity elements

$$A \cup \emptyset = A$$
$$A \cap E = A$$

5. Inverse elements

$$A \cup A^c = E$$
$$A \cap A^c = \emptyset$$

1. Associative properties

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$
$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

2. Commutative properties

$$p \vee q \equiv q \vee p$$
$$p \wedge q \equiv q \wedge p$$

3. Distributive properties

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

4. Identity elements

$$p \vee \phi \equiv p$$
$$p \wedge \tau \equiv p$$

5. Inverse elements

$$p \vee \rceil p \equiv \tau$$
$$p \wedge \rceil p \equiv \phi$$

Other properties

Let $A, B \subseteq E$.

- Absorption properties
  $$E \cup A = E, \quad \emptyset \cap A = \emptyset$$
- Simplification properties
  $$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$
- Idempotent properties
  $$A \cup A = A$$
  $$A \cap A = A$$
- De Morgan's laws
  $$(A \cup B)^c = A^c \cap B^c$$
  $$(A \cap B)^c = A^c \cup B^c$$
- Double complement property
  $$(A^c)^c = A$$

Sets vs. propositional forms

1. Absorption property
$$E \cup A = E, \quad \emptyset \cap A = \emptyset$$
2. Simplification property
$$A \cup (A \cap B) = A,$$
$$A \cap (A \cup B) = A$$
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$$(A \cup B)^c = A^c \cap B^c$$
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5. Double complement property
$$(A^c)^c = A$$

1. Absorption property
$$\tau \vee p \equiv \tau, \quad \phi \wedge p \equiv \phi$$
2. Simplification property
$$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$$
3. Idempotent properties
$$p \vee p \equiv p$$
$$p \wedge p \equiv p$$
4. De Morgan's laws
$$\rceil (p \vee q) \equiv \rceil p \wedge \rceil q$$
$$\rceil (p \wedge q) \equiv \rceil p \vee \rceil q$$
5. Double negation property
$$\rceil (\rceil p) \equiv p$$

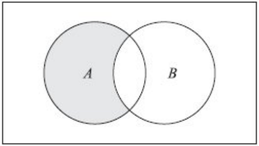
Difference of sets

Definition (Difference of sets)

Given two sets $A, B \subseteq E$ we called **the difference set of** A **minus** B to the set

$$A - B = \{x \in E \mid x \in A \ \wedge \ x \notin B\},$$

that is, it is the set of all elements in A that do not belong to B.



From the definition we get directly that

$$A - B = A \cap B^c$$

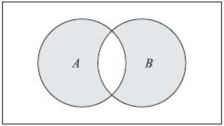
Symmetric difference of two sets

Definition (Symmetric difference)

Given two sets $A, B \subseteq E$ it is called **symmetric difference of A and B** to the set:

$$A \Delta B = (A - B) \cup (B - A),$$

that is, this is the set of all elements of the universal set E that belong A but do not belong to B, or conversely, that belong to B but does not belong to A (dark zone in the Venn diagram).



Cartesian product

Definition (Cartesian product)

Given two sets X and Y, we called the **cartesian product** of X and Y, and it is denoted as $X \times Y$, to the set of ordered pairs (x, y) such that $x \in X$ and $y \in Y$. That is:

$$X \times Y = \{(x, y) \mid x \in X \ \wedge \ y \in Y\}.$$

Example

If $A = \{a, b, c\}$ and $B = \{1, 2\}$, then
$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$ and
$B \times B = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

Cartesian product of several sets

Definition (Generalized cartesian product)

It is called the **cartesian product** of n sets $A_1, A_2, \ldots, A_n$ to the set $A_1 \times A_2 \times \cdots \times A_n$ that consists on all the n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in A_i$ for all $i = 1, 2, \ldots, n$. That is:

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i \ \forall i\}$$

Example

If $A = \{a, b, c\}$, $B = \{1, 2\}$ and $C = \{6, 7\}$, then
$A \times B \times C = \{(a,1,6),(a,2,6),(b,1,6),(b,2,6),(c,1,6),$
$(c,2,6),(a,1,7),(a,2,7),(b,1,7),(b,2,7),(c,1,7),(c,2,7)\}$

Coverings

Definition (Covering)

We call that a family of subsets $\{A_i \mid i \in I\}$ of E is a **covering** of a set B if

$$B \subseteq \bigcup_{i \in I} A_i.$$

In another way, a covering of B is a **set that consists on sets** whose union **contains** B.

In this definition the set I can be either finite or infinite.

Example

Let $A_1 = \{1,2,3\}$, $A_2 = \{2,3,4\}$ and be the sets $A_3 = \{4,5,6,7\}$. Then $\{A_1, A_2, A_3\}$ is a covering of the set $B = \{x \in \mathbb{N} \mid 1 \leq x \leq 5\}$. But not of the set $C = \{1,2,3,7,8\}$.

Partitions

Definition (Partition)

We say that a family of subsets $\{A_i \mid i \in I\}$ of E is a **partition** of the set B if

$$B = \bigcup_{i \in I} A_i$$

and the sets A_i are **pairwise disjoint** (that is, $A_i \cap A_j = \emptyset$ si $i \neq j$).

In fact, every partition is a covering.

Example

Consider the sets $A_1 = \{1,2,3\}$, $A_2 = \{4,5,6\}$ and $A_3 = \{7,8,9,10\}$. The family of sets $\{A_1, A_2, A_3\}$ is a partition of the set $B = \{x \in \mathbb{N} \mid 1 \leq x \leq 10\}$, but it si not a covering of the set $C = \{x \in \mathbb{N} \mid 1 \leq x \leq 8\}$.

Correspondence

Definition (Correspondence)

Given two sets A and B, we called a **correspondence** between A and B to a matching of elements of A with elements of B. A is the **initial set** and B is the **final set**.

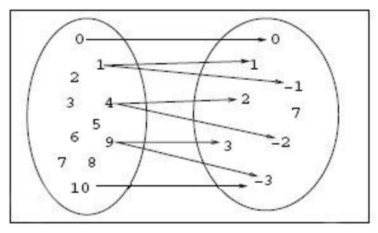
Correspondences are usually denoted in the form $f : A \to B$. If an element of $a \in A$ has associated another element $b \in B$, we say that b is an **image** of a, or that a is a **preimage** de b. Let us see some related terms.

- **f(a)**: Set of images of a.
- $\mathbf{f^{-1}(b)}$: Set of preimages of b.
- **Domain** of **f**: $Dom(f) = \{a \in A \mid f(a) \neq \emptyset\}$.
- **Range** or **image** of **f**: $\mathrm{Ran}(f) = \mathrm{Im}(f) = f(A) = \{b \in B \mid f^{-1}(b) \neq \emptyset\}$.
- **Graph** of **f**: $\mathrm{Graph}(f) = \{(a, b) \in A \times B \mid b \in f(a)\}$.

A correspondence is perfectly defined from its graph. In some cases it is given as the definition of the correspondece.
A **correspondence** is a subset of the cartesian product $A \times B$.

Example

Let $A = \{0,1,2,3,4,5,6,7,8,9,10\}$ and $B = \{0,1,-1,2,-2,3,-3,7\}$. Consider the correspondence $f : A \to B$ given by:



That is $f(1) = \{-1,1\}$, $f^{-1}(0) = \{0\}$, $f^{-1}(-3) = \{9,10\}$, $f^{-1}(7) = \emptyset$, $f(3) = \emptyset$, $\mathrm{Dom}(f) = \{0,1,4,9,10\}$, $\mathrm{Im}(f) = \{0,1,-1,2,-2,3,-3\}$ and $\mathrm{Graph}(f) = \{(0,0),(1,1),(1,-1),(4,2),(4,-2),(9,3),(9,-3)(10,-3)\}$.

Inverse correspondence

Definition (Inverse correspondence)

Given a correspondence $f : A \to B$ we called the **inverse correspondence of** f to $f^{-1} : B \to A$ whose associated graph is

$$\text{Graph}(f^{-1}) = \{(b,a) \in B \times A \mid (a,b) \in \text{Graph}(f)\}.$$

Example

The inverse correspondence of the previous example is the one whose graph is

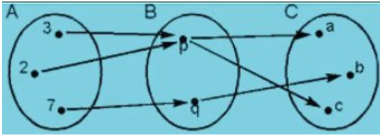
$$\text{Graph}(f^{-1}) = \{(0,0),(1,1),(-1,1),(2,4),(-2,4),(3,9),(-3,9),(-3,10)\}.$$

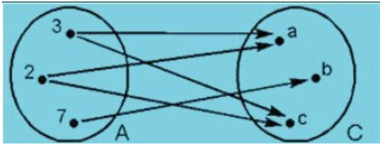
Composition of correspondences

Definition (Composition of correspondences)

Let $f : A \to B$ and $g : B \to C$ be two correspondences, with respective graphs $F \subseteq A \times B$ and $G \subseteq B \times C$. It is defined the **composition of** g **and** f as the correspondence $g \circ f : A \to C$ such that $(g \circ f)(a) = g(f(a))$ for all $a \in A$. In other words, it is the correspondence whose graph is:

$$\{(a, c) \in A \times C \mid \exists b \in B \text{ con } (a, b) \in F \wedge (b, c) \in G\}$$





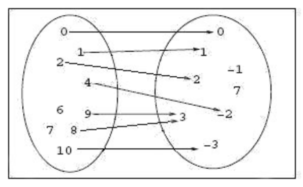
Function

Definition (Function)

It is said that a correspondence $f : A \to B$ is a **function** (or a map, or a mapping) if every element of A has one image **at most**.

Example

The following correspondence is a function



Injective function

Definition (Injective map)

A function $f : A \to B$ is said to be **injective** when all elements in A have different images, that is, if the following condition is satisfied

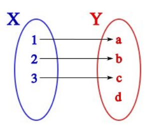
$$\forall a_1, a_2 \in A \ (a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)),$$

or the equivalent condition

$$\forall a_1, a_2 \in A \ (f(a_1) = f(a_2) \Rightarrow a_1 = a_2).$$

Example

The following function from X to Y is injective:



Surjective function

Definition (Surjective function)

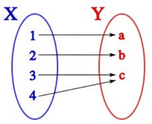
A function $f : A \longrightarrow B$ is said to be **surjective** when all elements in B have some preimage, that is, the following condition is satisfied

$$\forall b \in B \;\; \exists a \in A \text{ such that } f(a) = b,$$

or equivalently $Im(f) = B$.

Example

The following function from X to Y is surjective:



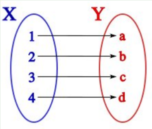
Bijective function

Definition (Bijective dunction)

A function $f : A \to B$ is said to be a **bijective** when it is injective and surjective.

Example

The following function from X to Y is bijective:

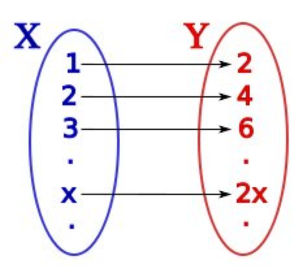


A bijection between two infinite sets

Infinite subsets can be as bigger as the whole set!

A bijection between two infinite sets

Take the sets $X = \mathbb{N}$ of all natural numbers and the set $Y = \{2x \mid x \in \mathbb{N}\}$ of all even natural numbers. We have that $Y \subsetneq X$ (Y is *strictly contained* in X, that is, it is contained in Y but it is not equal to Y). However, there exists a bijection between X and Y!

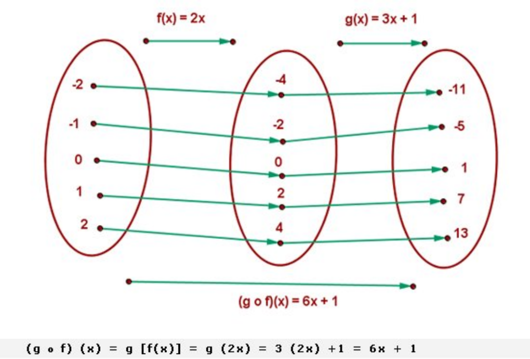


Composition of maps

2 maps can be composed

If $f : A \to B$ and $g : B \to C$ are 2 maps, with g surjective, then the composition $g \circ f : A \to C$ is also a map.

An example of a composition of 2 maps:



Properties of the composition of functions

Definition

Given a set A, we define the **identity function** on A as the function $id_A : A \longrightarrow A$ such that $id_A(a) = a$ for all $a \in A$.

Proposition

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions.

1. If f and g are injective, then $g \circ f$ is also injective.
2. If f and g are surjective, then $g \circ f$ is also surjective.
3. If f and g are bijective, then $g \circ f$ is also bijective.
4. f is bijective if and only if the inverse correspondence f^{-1} is a map.
5. f is bijective if and only if there exists another function $h : B \longrightarrow A$ such that $h \circ f = id_A$ and $f \circ h = id_B$. Moreover, inthis case we have $h = f^{-1}$.