DISCRETE MATHEMATICS PROBLEMS

Lesson 2.1 Sets

1. Write the following statements with the notation of set theory:

   a) x is an element of the set A.
   b) a does not belong to A.
   c) B is a subset of A.
   d) D is included in C.
   e) E is not a subset of F.
   f) A is a set without elements.
   g) B is contained in C and C is contained in B.

2. Give an extensional definition of the following sets:

   a) $\{x \in \mathbb{N} \mid x < 9\}$.
   b) $\{x \in \mathbb{N} \mid 2x^2 - 3x + 1 = 0\}$.
   c) $\{x \in \mathbb{Q} \mid 2x^2 - 3x + 1 = 0\}$.
   d) $\{x \in \mathbb{N} \mid x = 2n + 1, n \in \mathbb{N}\}$.

3. Give an intensional definition of the following sets:

   a) $\{2, 3, 4, 5, 6, 7\}$.
   b) $\{2, 4, 6, 8, \ldots\}$.
   c) $\{1, 3, 5, 7, \ldots\}$.
   d) $\{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$.

4. Let $A = \{1, 2, 3, 4\}$. Say which of the following statements are right and which are false. If someone statment is false, explain why it is false.

   a) $\{1, 4\} \subseteq \wp(A)$
   b) $4 \in A$
   c) $\{4\} \in \wp(A)$
   d) $\{1, 4\} \subseteq A$
   e) $4 \subseteq A$
   f) $4 \in \wp(A)$
   g) $\{4\} \subseteq A$
   h) $\{4\} \in A$
   i) $\emptyset \in \wp(A)$

5. If A and B are arbitrary sets of a set E, complete the following statements filling the gaps with the symbols $\subseteq$, $\supseteq$, or NC (Non comparable) between each pair of sets:

   a) $\emptyset \ldots A$
   b) $A \ldots A \cap B$
   c) $B \ldots B \setminus A$
   d) $B \ldots A^c \cup B$
   e) $A \setminus B \ldots B^c$
   f) $B^c \ldots B \setminus A$
   g) $A \cap B \ldots A^c \cup B$
   h) $A \setminus B \ldots B^c$
   i) $A \ldots A \cup B$

6. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$, and $C = \{3, 4, 5, 6\}$ be sets. Consider their complementary sets respect to the set $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find:

a) $(A \cap B) \cup C$
b) $(A \cup C) \cap (B \cup C)$
c) $A^c \cup B^c$
d) $(A \cap B)^c$
e) $A \times C$
f) $C \times A$
g) $(A \cup B) \setminus C$
h) $A \setminus (B \cap C)$
i) $E \setminus (A \cup B \cup C)$

7. If A, B, and C are arbitrary sets of a set E, prove:

   a) $A^c \setminus B^c = B \setminus A$
   b) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
   c) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$
   d) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$

8. If A, B, and C are arbitrary sets, say which of the following statements are right justifying your answer:

   a) Si $A = B \setminus C$, then $B = A \cup C$
   b) $(A \cup B) \setminus B = A$
   c) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
   d) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$

9. Simplify the following expressions using the properties of the algebra of sets:

   a) $(A^c \cap B^c) \cap [((A \cup B) \cap (A \cup B^c)) \cup ((A \cap B) \cup (A^c \cap B))]$
   b) $((A \cap (B \cup C)) \cap (A \setminus B)) \cap (B \cup C^c)$

10. Given two sets A and B, we define the **symmetric difference** of A and B, denoted as $A\Delta B$, as the set
$$A\Delta B = (A \setminus B) \cup (B \setminus A).$$
    Show the following properties concerning the symmetric difference:

    a) $A\Delta B = (A \cup B) \setminus (A \cap B)$
    b) $\emptyset \Delta B = B$
    c) $B\Delta B = \emptyset$
    d) $A\Delta B = B\Delta A$

11. Given the sets $E = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 4\}$, and $B = \{1, 4, 6\}$, find the subsets $R = A \setminus B$, $S = B \setminus A$, $V = A \cap B$, and $W = E \setminus (A \cup B)$. Is $\{R, S, V, W\}$ a partition of the set E?

12. Prove that the family of sets $\{A_k, k \in \mathbb{N}^*\}$[1], defined by:
$$A_k = \{2k - 1, 2k + 1\}, k = 1, 2, \dots$$
    is a covering of the set of odd natural numbers, but it is not a partition.

13. Let g be the correspondence from $\mathbb{N}$ to $\mathbb{N}$ defined by $G = \{(x, y) : x + 2y = 12\}$.

[1] The * means that the 0 is not included.

a) Give an extensional definition of the graph of this relation.

b) Find the domain and the range of this relation.

c) Give an extensional definition of the graph of the inverse relation g^{-1}.

d) Find $G \circ G$.

14. *a*) Consider the set $V = \{1, 2, 3, 4\}$. From the following sets of ordered pairs of $V \times V$, say which ones define maps from V to V.

    (i) $F = \{(3, 1), (4, 2), (1, 1)\}$

    (ii) $G = \{(2, 3), (1, 6), (4, 2), (3, 4)\}$

    b) Which conditions are necessary in order to have that the set of ordered pairs $F = \{(1, 5), (3, 1), (4, 7), (-2, -3)\}$ defines a map from the set A on the set B?

15. *a*) Consider the sets $X = \{2, 4, 5\}$ and $Y = \{1, 2, 4, 6\}$. Among the following sets of ordered pairs of $X \times Y$, say which ones define maps from X to Y.

    (i) $F = \{(2, 4), (4, 1), (5, 6), (4, 2)\}$

    (ii) $G = \{(2, 4), (4, 6), (5, 1)\}$

    (iii) $H = \{(2, 6), (4, 6), (5, 1)\}$

    (iv) $J = \{(2, 2), (4, 4)\}$

    b) For the ones which are maps, say if they are injective, surjective, and/or bijective.

16. Find the graphs of the following maps:

    a) The map f defined from the set $M = \{1, 2, 3, 4, 5\}$ on the set $\mathbb{R}$ as

    $$f(x) = x^{02} + 2x - 1.$$

    b) The map f defined from the set $W = \{1, 2, 3, 4\}$ on the set $\mathbb{R}$ by the formula $f(x) = x^3$.

    c) The map g defined from the set $S = \{a, e, i, o, u\}$ on the set $A = \{a, b, \dots, y, z\}$ that joins every element of S with the letter that follows it in the alphabet.

17. Let $A = \mathbb{R} \setminus \{3\}$ and $B = \mathbb{R} \setminus \{1\}$. Consider the function f from A to B given by

    $$f(x) = \frac{x-2}{x-3}.$$

    Verify that it is injective and surjective. Find a formula to define f^{-1}.

18. *a*) Given a real function $f(x) = \frac{2x}{1+x}$, give its domain and find its inverse function wherever it has sense.

    b) Let f be the real one variable function given by $f(x) = \frac{x+1}{2x-1}$. Find its domain and test that $f \circ f$ is the identity. Find the inverse function f^{-1} wherever it has sense.

19. Let f and g be the real one variable functions given by:

    $$f(x) = x^2, g(x) = \sqrt{\frac{x}{x-1}}.$$

a) Compute $h = f \circ g$ and give its domain.

b) Compute h^n, being n every natural number greater than 1.

20. Let $f_{a,b}$ be an map from $\mathbb{R}$ to $\mathbb{R}$ defined as $f_{a,b}(x) = ax + b$ with $a \neq 0$ and $a, b \in \mathbb{R}$. Prove that $f_{a,b}$ is bijective and compute its inverse map.