DISCRETE MATHEMATICS PROBLEMS

Lesson 2.2 Relations

1. Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$, and let us consider the following binary relation R between A and B:

$$R = \{(a, 1), (a, 3), (b, 2), (b, 5), (d, 1), (d, 4), (d, 5)\}$$

   a) Find the matrix of R.

   b) Find the domain and the image of R.

   c) Find the inverse relation of R, that is R^{-1}.

   d) Find $R \circ R^{-1}$ and $R^{-1} \circ R$.

2. Let A and B be the sets of problem 1. Let S be the binary relation from B to A whose matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

   a) Give an extensional or an intensional definition of the relation S. Give also its domain and its range.

   b) Find the relation $R \circ S$ and $S \circ R$, being R the binary relation of problem 1.

3. Let $X = \{1, 2, 3, 4, 5, 9\}$. Let R be the binary relation on X given by

$$xRy \leftrightarrow y = x^2.$$

   a) Give the extensional definition of the relation R.

   b) Obtain the relations $R \circ R$ and $R^{-1} \circ R^{-1}$.

4. In the set $A = \{a, b, c, d\}$ let us consider the following relations:

$$R_1 = \{(a, b), (b, c), (a, c), (b, a)\}$$
$$R_2 = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, a)\}$$
$$R_3 = \{(a, a), (c, c), (a, c), (c, b), (a, b)\}$$
$$R_4 = \{(a, a), (b, b), (c, c), (d, d)\}.$$

   Represent them with a graph and with a matrix . Say also which properties do they verify.

5. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$. Give the extensional definition of each one of the following binary relations defined on A, and say which properties do they verify.

   a) $aRb \Leftrightarrow b$ is a multiple of a

   b) $aRb \Leftrightarrow a + b \leq 7$

c) $aRb \Leftrightarrow |a-b| = 2$

6. Let $A = \{a, b, c, d, e\}$ and let R be the binary relation defined on A whose matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   a) Prove that R is an ordered binary relation. Is it a total order?
   b) Represent R with a Hasse diagram.

7. Let us consider the divisibility relation on $A = \{2, 3, 4, 6, 9, 12, 18, 36, 72\}$.

   a) Represent the ordered set A with a Hasse diagram.
   b) Find (if there are some) the minimal and maximal elements, and the maximum and minimum of A.
   c) Find (if there are some) the minimal and maximal elements, and the maximum and minimum of the subsets $B = \{2, 3, 6, 12, 18\}$, $C = \{4, 9, 12, 36\}$, $D = \{3, 12, 18\}$, $E = \{4, 6\}$ and its bounds respect to A

8. Let $A = \{a, b, c\}$ and let us consider the set $(\mathcal{P}(A), \subseteq)$, being $\mathcal{P}(A)$ the power set of A. Draw the Hasse diagram of this relation and find the minimal and maximal elements, and the maximum and minimum of the set $\{\{b\}, \{c\}, \{b, c\}\}$ as a subset of $\mathcal{P}(A)$.

9. Give an example of an ordered set with a subset of two elements that was upper bounded but without supremum.

10. The quotient set of a binary relation R on the set $A = \{1, 2, 3, 4, 5, 6\}$ is $A/R = \{\{2\}, \{1, 3, 6\}, \{4, 5\}\}$. Find the matrix associated to the binary relation R.

11. On the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ the following relation is given:

    xRy if and only if x and y has the same number of divisors.

    Show that R is an equivalence binary relation and compute its quotient set.

12. On the set $\mathbb{Z}$ of integer numbers the following relation is defined:

$$xRy \leftrightarrow x^2 - y^2 = x - y.$$

    Prove that R is an equivalence binary relation and compute the equivalence classes of 0 and 3. Say which is the equivalence class of an arbitrary integer number a.

13. On the set $\mathbb{R}$ of real numbers the following binary relation is defined:

$$xRy \leftrightarrow x - y \in \mathbb{R}$$

    Prove that R is an equivalence binary relation and give the equivalence classes of 0 and 1/2.

14. On the set $\mathbb{Z}_6$ the following relations are defined:

$$xR_1y \leftrightarrow x = y \vee 5x = y$$

$$xR_2y \leftrightarrow x = y \vee 3x = y$$

Prove that R_1 is an equivalence binary relation and find its quotient set. Prove that R_2 is an order and give its Hasse diagram.