Lesson 4. Combinatorics

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Contents

Rules of combinatorics

Classical combinatorics is based on determining the number of configurations with given properties.

There are three fundamental rules in combinatorics:

- The addition rule.
- The multiplication rule.
- The pigeonhole principle.

The addition rule

The addition rule

Let $\{A_1, \ldots, A_n\}$ be a family of subsets that are pairwise disjoint, and

$$A = \cup_{i=1}^{n} A_i.$$

Then we have:

$$|A| = |A_1| + \ldots + |A_n| = \sum_{i=1}^{n} |A_i|.$$

Here $|A|$ represents the cardinal of A. If A is finite, then it is the number of elements in A.

The inclusion-exclusion principle

The addition rule cannot be applied if the sets are not pairwise disjoint.

The following principle can be applied in these cases:

The inclusion-exclusion principle

Let $\{A_1, \ldots, A_n\}$ be a family of subsets. Then we have

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i,j \leq n}^{n} |A_i \cap A_j| \\ + \sum_{1 \leq i,j,k \leq n} |A_i \cap A_j \cap A_k| \\ + \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n| \end{aligned}$$

The multiplication rule

The multiplication rule

Let us compute the number of n-tuples of the form $(x_1, x_2, \ldots, x_n)$.

Suppose that the element x_i can be chosen in exactly α_i ways,

and for each $i = 1, 2 \ldots, n-1$ the element x_{i+1} can be chosen in α_{i+1} which does not depend on the choice of the elements $x_1, \ldots, x_i$.

Then the number N of all such ordered n-tuples satisfies

$$N = \alpha_1 \cdot \alpha_2 \cdots \alpha_n.$$

The pigeonhole principle

The pigeonhole principle (or Dirichlet's principle)

If at least $nk+1$ objects (or pigeons) are distributed among k boxes (or pigeonholes), then some box must contain at least $n+1$ objects.

Corollary

If at least n objects (or pigeons) are distributed among k boxes (or pigeonholes), with $n > k$ then some box must contain at least $[n/k]$ objects.

Permutations, variations, and combinations

The following formulas can be obtained applying the multiplication rule in a set of different elements:

- Permutations (without repetition)
- Variations (with or without repetition)
- Combinations (without repetition)

Permutations without

Definition (Permutation)

A **permutation** in a set A is a particular arrangement of all their elements where the order is important.

How to compute the total number of permutations in a set where all their elements are different?

Let A be a finite set with n different elements. The number $P(n)$ of different arrangements of the n elements of A is

$$P(n) = n!$$

Variations without repetition

Definition (Variation without repetition)

A **variation without repetition** of k elements in a set with n different elements is a particular arrangement of these k elements, where these elements **cannot be repeated** and the order of the elements is important.

How to compute the total number of variations of k elements inside a set of n elements **without repetition**?

Let $V(k,n)$ denote the number of all k-element variations from n elements **without repetition**. Then

$$V(n,k) = \frac{n!}{(n-k)!} = n\cdot(n-1)\cdot(n-2)\dots(n-k+1).$$

Variations with repetition

Definition (Variation with repetition)

A **variation with repetition** of k elements in a set with n different elements is a particular arrangement of k elements from these n, where these k elements **can be repeated** and the order is important.

How to compute the total number of variations of k elements inside a set of n elements **with repetition**?

Let $VR(k,n)$ denote the number of all k-element variations from n elements **with repetition**. Then

$$VR(n,k) = n^k.$$

Combinations without a repetition

Definition (Combinations without repetition)

Let $C(k,n)$ denote the number of all k-combinations from n-elements where the n elements are different and the order of the k elements that we have chosen is not important. Then

$$C(n,k) = \frac{n!}{(n-k)!k!}$$

The expression $\frac{n!}{(n-k)!k!}$ is the definition of the combinatorial number $\binom{n}{k}$

Summary

Summary of formulas

The following table show the different arrangements in a set where all the elements are different.

	Order?	**Repetition?**
Variations without	Yes	No
Variations with repetition	Yes	Yes
Combinations	No	No

Permutations are a particular case of variations without repetition.

Permutations and combinations in multisets

The following formulas can be obtained applying the multiplication rule in a set where we have serveral types of elements and some elements of each type. Such a set will be called a **multiset**:

- Permutations (with repetition)
- Combinations (with repetition)

Permutations with repetition

Definition (Permutations with repetition)

Suppose that the elements of a set A are of m different classes $x_1, x_2, \ldots, x_m$, and we have α_i elements of the class x_i with $|A| = \alpha_1 + \ldots + \alpha_m = n$.

The number of possible permutations of this n elements is

$$P_{\alpha_1,\ldots,\alpha_m}(n) = \frac{n!}{\alpha_1! \cdots \alpha_m!}$$

Combinations with repetition

Definition (Combinations with repetition)

Suppose that we have a set A of n-types (plenty of each type) and we want to consider the combinations of k-elements from these n-types. Two combinations are equal if they have the same type of elements and each one is repeted the same number of times. This is denoted as

$$CR(n,k) = \frac{(k+n-1)!}{k!(n-1)!}$$