Lesson 2.2 Relations

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Contents

1. Binary relations
   - Definition and notation
   - Graphic representation
   - Matrix representation
2. Operations with relations
   - Inverse relation
   - Composition of relations
3. Binary relations on a set
   - Properties
   - Closures
4. Binary relations of order
   - Partial and total orders
   - Hasse diagram
   - Maximum, minimum, maximal and minimal elements
   - Bounds
5. Equivalence relations
   - Equivalence classes and quotient set
   - Congruence relations

Binary Relations

Definition (Binary Relation)

A **binary relation** R between the sets A and B is a subset

$$R \subseteq A \times B.$$

Since two sets A and B are involved we say that its **degree** is 2.

R is understood as a **relation** between the elements of A and the elements of B. In other words, R can be seen as the graph of a correspondence from A to B.

The case of binary relations is of particular interest, because of the richness of its results and the techniques.

Notation

If R is a relation between A and B, the fact that an ordered pair (a, b) belongs to R can also be denoted as aRb. On the other hand, the contrary, that is $(a, b) \notin R$, can be denoted as $a \not R\, b$.

Examples

Example (Examples of binary relations)

- If $A = \{1,2,3,4\}$ and $B = \{a,b,c,d\}$ we can define the following binary relation between A and B: $R = \{(1,b),(1,c),(2,a),(3,a),(3,b)\} \subseteq A \times B$. So that, $1Rb$, $1Rc$, $2Ra$, $3Ra$, $3Rb$ and $4 \not R\, x \ \forall x \in B$.
- If $A = B = \mathbb{N}$, we can define the binary relation R between A and B as:

  $$aRb \leftrightarrow a \textbf{ is a divisor of } b$$

  for every pair $(a,b) \in \mathbb{N} \times \mathbb{N}$. That is,

  $$R = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ is a divisor of } b\}.$$

  Usually, the condition a **is a divisor of** b is denoted as $a \mid b$. For example, we have that $3R6$ (or $(3,6) \in R$) and that $7 \not R\, 15$ (or $(7,15) \notin R$)).

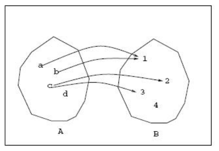
Notation:

When R is a binary relation between A and A, we will simply say that R **is a binary relation on** A.

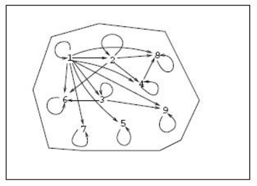
Graphic representation

Let us see how to represent a relation:

- Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$ and the relation $R = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$. Looking at R as a correspondence from A to B we can represent it using Venn diagrams as:



- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and consider the divisibility relation on A, that is, $aRb \leftrightarrow a \mid b$. This can be represented graphically using arrows to join the pairs of elements a and b of A such that aRb:



Matrix representation

Matrix representation

A binary relation can be represented by a matrix whenever the sets that are involved in the relation were finite. Suppose that $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_p\}$. Then, the associated matrix to R is the boolean matrix (that only consists on ones and zeros) with m arrows and p columns

$$M_R = \begin{pmatrix} r_{11} & \cdots & r_{1p} \\ \vdots & \vdots & \vdots \\ r_{m1} & \cdots & r_{mp} \end{pmatrix} \text{ with } r_{ij} := \begin{cases} 1 & \text{if} \quad a_i R\, b_j \\ 0 & \text{if} \quad a_i \not R\, b_j \end{cases}$$

Example (Consider the sets $A = \{2,3,5\}$ and $B = \{4,6,9,10\}$, and the relation)

$$R := \{(2,4),(2,6),(2,10),(3,6),(3,9),(5,10)\} \subseteq A \times B$$

(that is, aRb if and only if $a \mid b$). Then the associated matrix to R is

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operations with relations

As a binary relation is a subset of the set $A \times B$, given two relations R and S between them, we can define the following operations: $R \cup S$, $R \cap S$, R^c , and $R - S$.

We can also refer to the notions of **domain** and **image** of the relation R (denoted as $\mathrm{Dom}(R)$ and $\mathrm{Im}(R)$, respectively):

$$\mathrm{Dom}(R) = \{a \in A \mid \exists b \in B, (a,b) \in R\},$$
$$\mathrm{Im}(R) = \{b \in B \mid \exists a \in A, (a,b) \in R\}$$

Definition (Inverse relation of R)

$$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\},$$

that is a relation from B to A.

Operations with relations

Definition (Composition of two relations)

If R is a binary relation between A and B, and S is a binary relation between B and C, we can consider them as correspondences and define the composition $S \circ R$ as the graph of the composition of both of them, that is:

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \text{ with } (a, b) \in R \ \wedge \ (b, c) \in S\}.$$

As we can see, $S \circ R$ is a binary relation from A to C.

Properties of binary relations on a set

Among the properties that can (or cannot) be verified on a binary relation R on a set A, the most important are the following ones:

- Reflexive: if $\forall x \in A,\ xRx$
- Symmetric: if $\forall x, y \in A,\ xRy \longrightarrow yRx.$
- Antisymmetric: if $\forall x, y \in A,\ (xRy) \wedge (yRx) \longrightarrow x = y.$
- Transitive: if $\forall x, y, z \in A,\ (xRy) \wedge (yRz) \longrightarrow (xRz).$

Characterizations of these properties

Theorem

Characterization of these properties Let $R \subseteq A \times A$ and let M be the associated matrix R (when A is finite).
Let $\Delta = \{(x,x) \mid x \in A\}$ be the **diagonal relation** *on $A \times A$. Then*

- *R is reflexive $\leftrightarrow (\Delta \subseteq R) \leftrightarrow M$ has $1'$s in all position of the main diagonal.*
- *R is symmetric $\iff (R = R^{-1}) \leftrightarrow (M = M^t)$, that is, if M is symmetric.*
- *R is antisymmetric $\leftrightarrow (R \cap R^{-1} \subseteq \Delta) \leftrightarrow (M \wedge M^t \leq Id) \leftrightarrow$ if there are no two symmetric positions outside the main diagonal whose values were 1 simultaneously. Here, I denotes the* **identity matrix**, *that is, the one that has ones in the main diagonal and zeros outside of it.*
- *R is transitive $\leftrightarrow (R \circ R \subseteq R)$.*

Closure of binary relations on a set

In some cases a relation R maybe does not verify the reflexive, symmetric, or transitive properties. However, in these cases we can find a relation R' that contains R, is the smallest possible that contains R and is reflexive (resp. symmetric or transitive). This new relation is called the **reflexive (resp. symmetric or transitive) closure** of R.

Definition (Closures)

Let R be a binary relation on a set A.

- We call **reflexive closure** of R, denoted by $r(R)$, to the smallest reflexive relation containing R.
- Let R be a binary relation on a set A. We call **symmetric closure** of R, denoted by $s(R)$, to the smallest symmetric relation containing R.
- Let R be a binary relation on a set A. We call **transitive closure** of R, denoted by $t(R)$, the smallest transitive relation containing R.

A relation that coincides with some closure verifies the corresponding property.

In order to compute the transitive closure, we compute R^2. If $R \circ R \subseteq R$ it is transitive we are done. If not, we apply again the characterization again with $R \circ R$ instead of R and so on. We will always arrive to the get the transitive closure, since A has a finite number of elements.

Binary relations of order

Definition (Order)

A binary relation R on a set A is an **order** on A if it is reflexive, antisymetric, and transitive.

Definition (Total order)

A binary relation R on A is a **total order** on A if $\forall x, y \in A, \quad (xRy) \vee (yRx)$.

Definition (Partial order)

If a relation is not a total order, the we say that it is a **partial order**.

Example (As typical examples of relations of order, we can cite)

- the inclusion of sets, $\subseteq$;
- the inequality between numbers, $\leq$;
- the lexicographic order between words, and
- the divisibility relation between natural numbers.

Hasse diagram

Hasse diagram

Let us consider a binary relation R with an order defined on a set A. We can represent the relation graphically with a diagram that we will construct as follows:

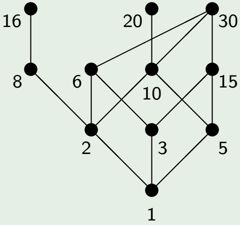
- If aRb, we will write a below b and we will draw a line from a to b, whenever a and b were different. Since the relation is reflexive, the pairs of the form (x, x) are not represented.
- We do not draw the lines that can be deduced by transitivity: that is, if aRb and bRc, we draw a line from a to b and another one from b to c, but we do not draw a line from a to c.

This diagram is called the **Hasse diagram** of the relation on A.

Example: Divisibility relation

Example (A Hasse diagram)

Let $A = \{1, 2, 3, 5, 6, 8, 10, 15, 16, 20, 30\}$ and consider the divisibility relation on A (denoted by $|$). The Hasse diagram of this relation is represented with this figure:



We see that 2 is related with 30, because there is at least one upward path. On the other hand, 2 and 15 are not related, because there is not an upward path between these numbers.

Maximum, minimum, maximal and minimal elements

Let A be a set endowed with a relation of order $\preceq$.

Definition

- An element $m \in A$ is a **maximum** if $\forall x \in A$ we have $x \preceq m$.
- An element $m \in A$ is a **minimum** if $\forall x \in A$, we have $m \preceq x$.
- An element $m \in A$ is **maximal** if

$$\forall x \in A \quad (m \preceq x \to m = x),$$

  that is, if there is not any other element of A that comes after m.
- An element $m \in A$ is **minimal** if

$$\forall x \in A \quad (x \preceq m \to m = x),$$

  that is, if there is no other element of A that comes before m.

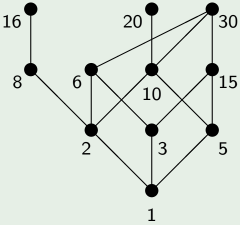
Bounds

Let A be a set endowed with a binary relation of order $\preceq$, and let B be a subset of A.

- It is said that $a \in A$ is an **upper bound** of B if $\forall x \in B$, $x \preceq a$. If B has upper bounds, then it is said that B is **upper bounded**.
- It is said that $a \in A$ is a **lower bound** of B if $\forall x \in B$, $a \preceq x$. If B has lower bounds, then it is said that B is **lower bounded**.
- It is said that $a \in A$ is the **supremum** of B ($\sup B$) if a is the smallest upper bound of B (that is, it is the minimum of the set of upper bounds of B).
- It is said that $a \in A$ is the **infimum** of B ($\inf B$) if a is the biggest lower bound of B (that is, it is the maximum of the set of lower bounds of B).

Example

Example (Let us go back to the previous example)



If we consider the set $B = \{2, 10, 5\}$, then teh upper bounds of A are 10, 20, and 30 and its supremum is 10. The only lower bound of B is 1, and therefore it is its infimum. On the other hand, the maximum of B is 10, B has no minimum, 10 is a maximal element, and the minimal elements of B are 2 and 5.

Definition (Equivalence relation)

A binary relation R on a set A is an **equivalence** relation if it is reflexive, symmetric, and transitive.

Definition (Equivalence class and quotient set)

If R is an equivalence relation, then we called the **equivalence class** of $a \in A$ respect to R to the set

$$[a] = \bar{a} = [a]_R := \{x \in A \mid aRx\}.$$

The set that consists on all the equivalence classes of the relation R is called the **quotient set** and it is denoted by A/R. That is

$$A/R := \{[a] \mid a \in A\},$$

Proposition

- *If R is an equivalence relation on the set A, then*

$$aRb \longleftrightarrow [a] = [b]$$

- *The* **quotient set** *is a partition of the set A.*

Example

Let us consider the set $A = \{1, 2, 3, 4, 5\}$ with the following equivalence relation $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (2, 4), (4, 2)\}$. We have that

$$[1] = [3] = \{1, 3\}$$

$$[2] = [4] = \{2, 4\}$$

$$[5] = \{5\}$$

Therefore, the quotient set is

$$A/R = \{[1], [2], [5]\}$$

Example: Congruence relations

Example (Example: Congruence relations)

Given a positive integer $m \in \mathbb{Z}_+$ we define (in the set of integers $\mathbb{Z}$) the following binary relation):

$$aRb \iff a - b \text{ es un multiple of } m.$$

- R is an equivalence relation called the **congruence relation modulo** m.
- The quotient set $\mathbb{Z}/R$ is dentoed by $\mathbb{Z}_m$.
- If $\bar{a}$ is the equivalence class of the integer $a \in \mathbb{Z}$, then

$$\mathbb{Z}_m = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{m-1}\}$$

- For this relation, instead of aRb we write

$$a \equiv b \pmod{m}$$

and we read *"a is congruent with b modulo m"*.